Non-perturbative solution of nonlinear Heisenberg equations

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Abstract

A new non-perturbative method of solution of the nonlinear Heisenberg equations in finite-dimensional subspace is illustrated. The method, being a counterpart of the traditional Schrödinger picture method, is based on a finite operator expansion into the elementary processes. It provides us with the insight into the nonlinear quantal interaction from the different point of view. Thus one can investigate the nonlinear system in both pictures of quantum mechanics.

1 Introduction

The use of laws of quantum mechanics in the description of nonlinear systems confronts us with the qualitatively new difficulties. Namely, to investigate their dynamics in the Heisenberg picture we have to solve the nonlinear operator equations, a task which is highly nontrivial even for the simplest systems. The difficulties are also encountered with in the Schrödinger picture once we try to solve the Schrödinger equation explicitly [1]. Since some nonlinear systems solvable analytically in classical domain become insoluble when are quantized one can suppose that they are simultaneous influence of intrinsic stochastic effects, originating from the incompatibility of some observables, and nonlinearity which make the behaviour of such systems very complex and thus difficult to describe analytically.

The time evolution of quantum systems can be studied with the help of widely used Schrödinger picture method based on the integration of set of linear differential equations for components of a state vector in the Fock basis [2],[3]. Unfortunately, the expansion into the Fock-state basis can be infinite for some states, e.g. for coherent state, yielding the infinite set of these equations. Because it is practically impossible to solve the infinite system of the equations, the method provides us with exact solutions only for states from some finite-dimensional subspace of the Hilbert state space. On the other hand, it is advantageous sometimes to calculate the evolution of particular observables in the framework of Heisenberg picture. The motivation of the present paper is to find the operator analogue of the Schrödinger picture method in the Heisenberg picture and to show the equivalence and deep relationship between them.

As an illustrative example we consider here the simple system composed of two harmonic oscillators which oscillate with frequencies ω and 2ω and which are

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described by the annihilation (creation) operators $\hat{a}_1(\hat{a}_1^\dagger)$ and $\hat{a}_2(\hat{a}_2^\dagger)$ obeying the standard boson type commutation rules

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0, \quad i, j = 1, 2. \tag{1}$$

Let the two oscillators interact nonlinearly according to the following interaction Hamiltonian (the second-harmonic generation process [4])

$$\hat{H} = -\hbar\Gamma\hat{a}_1^{\dagger 2}\hat{a}_2 + \text{h.c.}, \tag{2}$$

where Γ denotes the nonlinear coupling constant; the symbol $\hbar$ is reduced Planck constant and h.c. stands for the Hermitian conjugate term. Here and in the following we assume that the free evolution was eliminated by the appropriate unitary transformation.

Employing the commutation rules (1) one can directly prove the existence of the following integral of motion

$$\hat{N} = \hat{n}_1 + 2\hat{n}_2, \tag{3}$$

corresponding to the total energy of the system. Here the photon number operator $\hat{n}_j$ of the jth oscillator , $j = 1, 2$ has been introduced. The eigenvectors of the integral of motion (3) then provide us with the natural orthonormal and complete basis in which the expressions have simple form. They are easy to find and have the form

$$\{|N-2l, l\rangle, l = 0, 1, ..., \left[\frac{N}{2}\right], N = 0, 1, ...\} \tag{4}$$

with orthonormality condition

$$\langle N-2l, l|M-2k, k\rangle = \delta_{NM}\delta_{lk} \tag{5}$$

and the resolution of unity operator

$$\sum_{N=0}^{\infty}\sum_{l=0}^{[\frac{N}{2}]}|N-2l, l\rangle\langle N-2l, l| = \hat{1}, \tag{6}$$

where $|n_1, n_2\rangle$ is the Fock state having energy $\hbar\omega n_1 + 2\hbar\omega n_2$; N is the eigenvalue of (3), $[N/2]$ represents the greatest integer less or equal to $N/2$ and δ_{lk} is the Kronecker symbol. The Hilbert state space of our system can then be expressed as a direct sum

$$\mathcal{H} = \sum_{N=0}^{\infty}\oplus\mathcal{H}^{(N)} \tag{7}$$

of the invariant $[N/2]+1$-dimensional subspaces $\mathcal{H}^{(N)}$ spanned on the basis vectors $|N-2l, l\rangle, l = 0, 1, ..., [N/2]$ corresponding to the fixed eigenvalue N. Using the standard properties of the annihilation and creation operators of the harmonic oscillator

$$\hat{a}_i|n_i\rangle = \sqrt{n_i}|n_i - 1\rangle, \quad \hat{a}_i^\dagger|n_i\rangle = \sqrt{n_i+1}|n_i+1\rangle, \quad i = 1, 2 \tag{8}$$

and employing the condition (5) one can show that the Hamiltonian (2) is represented by the following block diagonal matrix

$$\begin{aligned}\langle N-2l, l|\hat{H}|M-2k, k\rangle &= -\hbar[\Gamma\sqrt{(l+1)(N-2l)(N-2l-1)}\delta_{k,l+1} \\ &\quad +\Gamma^*\sqrt{l(N-2l+2)(N-2l+1)}\delta_{k,l-1}]\delta_{NM},\end{aligned} \tag{9}$$

where the symbol '$*$' represents the complex conjugation.

2 Schrödinger picture

Let us first recall the results obtained with the help of the Schrödiger picture method when applied to our system. As is well-known the time evolution of the state vector is governed by the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle, \tag{10}$$

where Hamiltonian $\hat{H}$ is given in (2). Rewriting (10) into the basis (4) with the help of (6) and (9) we successively arrive at the infinite number of sets of linear differential equations

$$\begin{aligned}\frac{dC_{N,l}}{dt} &= i\Gamma\sqrt{(l+1)(N-2l)(N-2l-1)}C_{N,l+1} \\ &\quad + i\Gamma^*\sqrt{l(N-2l+2)(N-2l+1)}C_{N,l-1}\end{aligned} \tag{11}$$

for components $C_{N,l} \equiv \langle N-2l, l|\psi(t)\rangle$, where $N = 0, 1, ...$ and $l = 0, 1, ..., [N/2]$. Assuming, however, the initial state to be from the finite-dimensional subspace

$$\mathcal{H}_K = \sum_{N=0}^{K} \oplus \mathcal{H}^{(N)}, \tag{12}$$

it is sufficient to solve only $K+1$ such sets labelled by eigenvalues $N = 0, 1, ..., K$ each of them with $[N/2]+1$ equations. Particularly, for states belonging to the subspaces $\mathcal{H}_2$, the set (11) is of the form

$$\frac{dC_{0,0}}{dt} = \frac{dC_{1,0}}{dt} = 0, \quad \frac{dC_{2,0}}{dt} = i\sqrt{2}\Gamma C_{2,1}, \quad \frac{dC_{2,1}}{dt} = i\sqrt{2}\Gamma^* C_{2,0} \tag{13}$$

and can be solved analytically. An interesting result is obtained assuming the system to be in the state $|0, 1\rangle$ at the beginning of the interaction. The initial conditions for the set (13) are then $C_{0,0}(0) = C_{1,0}(0) = C_{2,0}(0) = 0$, $C_{2,1}(0) = 1$ and the solution of (10) reads

$$|\psi(t)\rangle = \sum_{N=0}^{2}\sum_{l=0}^{[\frac{N}{2}]} C_{N,l}(t)|N-2l, l\rangle = i\frac{\Gamma}{|\Gamma|}\sin(\sqrt{2}|\Gamma|t)|2,0\rangle + \cos(\sqrt{2}|\Gamma|t)|0,1\rangle. \tag{14}$$

Hence we obtain the following expressions for the mean number of energy quanta in oscillators 1 and 2 in state (14)

$$\langle\hat{n}_1(t)\rangle = 2\sin^2(\sqrt{2}|\Gamma|t), \quad \langle\hat{n}_2(t)\rangle = \cos^2(\sqrt{2}|\Gamma|t). \tag{15}$$

This non-classical oscillatory behaviour can be interpreted from the point of view of Schrödiger picture as being a manifestation of quantum interference effect (14).

3 Heisenberg picture

In this picture the operators $\hat{a}_1$ and $\hat{a}_2$ for the system of interest evolve according to the Heisenberg equations of motion

$$i\hbar\frac{d\hat{a}_j}{dt} = [\hat{a}_j, \hat{H}], \quad j = 1, 2, \tag{16}$$

which after substitution (2) into (16) and application (1) read

$$\frac{d\hat{a}_1}{dt} = 2i\Gamma\hat{a}_1^\dagger\hat{a}_2, \quad \frac{d\hat{a}_2}{dt} = i\Gamma^*\hat{a}_1^2. \tag{17}$$

It is also well-known that the operators $\hat{a}_1(t)$ and $\hat{a}_2(t)$ can be equivalently expressed as follows

$$\hat{a}_j(t) = \exp(\frac{i}{\hbar}\hat{H}t)\hat{a}_j(0)\exp(-\frac{i}{\hbar}\hat{H}t) = \sum_{k=0}^{\infty}\frac{1}{k!}\frac{d^k\hat{a}_j(0)}{dt^k}t^k, \quad j = 1,2, \tag{18}$$

where the exponential operators have been expanded and (16) has been used repeatedly.

The power series on the right hand side (R.H.S.) of (18) is a perturbative solution of the equations (17) and provides us with two important informations. As becomes clear from the following it is advantageous to work in the normal ordering of the operators in which all creation operators stand to the left from all annihilation operators. First, the operator part of the solution (18), given by derivatives of operators $\hat{a}_1(t)$ and $\hat{a}_2(t)$ at $t = 0$, cannot contain products of operators other than those of leading to the annihilation of one energy quantum from the corresponding oscillator. This can be proved by deriving the Heisenberg equations of motion (17) and using consequently the commutators (1) to obtain the normally ordered expressions. From now any such product of operators at $t = 0$ is called a *process* in the corresponding oscillator and the number of operators in the product is called an *order* of the process. Secondly, calculating the perturbative solution (18) to the sufficiently high order and rearranging its terms appropriately, one can see that the solution is of the form of finite sum of the processes multiplied by various polynomials in t, which can constitute the first few terms of power series of well-known functions (it can be verified at least for the first few processes). This different point of view to the standard perturbative solution [5] is the core of our *non-perturbative* method developed in the following text. Hence one can surmise, that going to the infinity in the iterative procedure, the solution of the Heisenberg equations of motion (17) is of the form of *infinite* sum of processes multiplied by some time dependent functions

$$\hat{a}_1(t) \quad = \quad \hat{a}_1 + f_1(t)\hat{a}_1^\dagger\hat{a}_2 + f_2(t)\hat{a}_1^\dagger\hat{a}_1^2 + f_3(t)\hat{a}_2^\dagger\hat{a}_1\hat{a}_2 + ..., \tag{19}$$

$$\hat{a}_2(t) \quad = \quad g_1(t)\hat{a}_2 + g_2(t)\hat{a}_1^2 + g_3(t)\hat{a}_1^\dagger\hat{a}_1\hat{a}_2 + g_4(t)\hat{a}_2^\dagger\hat{a}_2^2 + ... \tag{20}$$

where $\hat{a}_j \equiv \hat{a}_j(0)$, $j = 1,2$. The functions f_j and g_j are called *amplitudes* of the corresponding processes in the following text.

Substituting (19) and (20) into (17) and comparing the coefficients related to the same process, the amplitudes f_j and g_j can be determined as solutions of a system of ordinary differential equations. For example, the equations for amplitudes f_1 and g_1 together with the initial conditions read

$$\frac{d}{dt}f_1(t) = 2i\Gamma g_1(t), \quad \frac{d}{dt}g_1(t) = i\Gamma^* f_1(t), \quad f_1(0) = 0, \quad g_1(0) = 1 \tag{21}$$

and have the following solutions

$$f_1(t) = i\frac{\sqrt{2}\Gamma}{|\Gamma|}\sin(\sqrt{2}|\Gamma|t), \quad g_1(t) = \cos(\sqrt{2}|\Gamma|t). \tag{22}$$

Employing (19) and (20) the operators of the number of the energy quanta in the oscillators 1 and 2 are of the form

$$\hat{n}_1(t) \quad = \quad \hat{a}_1^\dagger\hat{a}_1 + 2\sin^2(\sqrt{2}|\Gamma|t)\hat{a}_2^\dagger\hat{a}_2 + ..., \tag{23}$$

$$\hat{n}_2(t) \quad = \quad \cos^2(\sqrt{2}|\Gamma|t)\hat{a}_2^\dagger\hat{a}_2 + ..., \tag{24}$$

where relations (1) and (22) have been used. It is worth noting, that contrary to the R.H.S. of (24) the second term in (23) is the quantum contribution originating from the commutator $[\hat{a}_1, \hat{a}_1^\dagger] = \hat{1}$. Considering as in the previous section the input state to be $|0,1\rangle$ state and assuming that the next terms represented by dots in (23) and (24) do not contribute, one obtains for the mean number of the energy quanta in the oscillators the expressions

$$\langle \hat{n}_1(t) \rangle = |f_1(t)|^2 = 2\sin^2(\sqrt{2}|\Gamma|t), \quad \langle \hat{n}_2(t) \rangle = |g_1(t)|^2 = \cos^2(\sqrt{2}|\Gamma|t), \tag{25}$$

which are identical with the results (15) obtained by means of the Schrödinger picture method. Notice, that this derivation illustrates not only the mathematical equivalence of both methods but also their difference when one tries to distinguish between the classical and quantum contributions.

Although one could look at the method just described as being a satisfactory method, let us recall the reader, that its conclusion (25) rests on two crucial assumptions which were not justified at all. First, we have assumed implicitly, when deriving (21), that the higher order processes do not affect the first order ones (cosequently we have obtaind the finite set of differential equations for amplitudes f_1 and g_1). Secondly, the mean numbers of energy quanta in state $|0,1\rangle$ given by (25) have been derived under the assumption that only explicitly given terms in (23) and (24) contribute. To show that this is really the case, we have to formalise and precise the Heisenberg picture method. This is done in the following section.

4 General method

The previous section provides us with an illustrative example, how one can treat the system (2) within the framework of Heisenberg picture on the intuitive basis. In the present section we try to justify the intuitive assumptions discussed above and to generalize this treatment to the arbitrary finite-dimensional subspace. This can be achieved by the suitable parametrization of the problem under discussion. To that aim let us rewrite the expansions (19) and (20) into the compact forms

$$\hat{a}_1(t) = \sum_{i,j,k,l=0}^{\infty} f_{ijkl}(t)(\hat{a}_1^\dagger)^i(\hat{a}_2^\dagger)^j \hat{a}_1^k \hat{a}_2^l, \tag{26}$$

$$\hat{a}_2(t) = \sum_{o,p,r,s=0}^{\infty} g_{oprs}(t)(\hat{a}_1^\dagger)^o(\hat{a}_2^\dagger)^p \hat{a}_1^r \hat{a}_2^s, \tag{27}$$

where

$$2l + k - 2j - i = 1 \tag{28}$$

and

$$2s + r - 2p - o = 2 \tag{29}$$

holds. There are two facts which can make the convenient parametrization easier to find. First, as in the Schrödinger picture we can employ the existence of the integral of motion (3). Secondly, the discussion in the previous section indicates that the order of the process is of importance. Therefore we put

$$N = k + 2l, \quad M = i + 2j, \quad m = k + l, \quad R = i + j + k + l, \tag{30}$$

for oscillator 1 and similarly

$$N = r + 2s, \quad M = o + 2p, \quad m = r + s, \quad R = o + p + r + s, \tag{31}$$

for oscillator 2. Thus each process is parametrized by the parametres $N(M)$ (representing the amount of the annihilated (created) energy in the process), m (the total number of the annihilated energy quanta) and R (the order of the process). Substituting (30) into (26) and eliminating N by means of (28) we arrive at the following expansion

$$\begin{aligned}\hat{a}_1(t) &= \sum_{M=0}^{\infty} \sum_{m=[\frac{M}{2}]+1}^{M+1} \sum_{R=[\frac{M+1}{2}]+m}^{M+m} f_{MmR}(t) \\ &\quad \times (\hat{a}_1^\dagger)^{2R-M-2m} (\hat{a}_2^\dagger)^{M+m-R} \hat{a}_1^{2m-M-1} \hat{a}_2^{M-m+1}. \end{aligned} \tag{32}$$

In the same way we obtain

$$\begin{aligned}\hat{a}_2(t) &= \sum_{M=0}^{\infty} \sum_{m=[\frac{M+1}{2}]+1}^{M+2} \sum_{R=[\frac{M+1}{2}]+m}^{M+m} g_{MmR}(t) \\ &\quad \times (\hat{a}_1^\dagger)^{2R-M-2m} (\hat{a}_2^\dagger)^{M+m-R} \hat{a}_1^{2m-M-2} \hat{a}_2^{M-m+2}. \end{aligned} \tag{33}$$

Substituting (32) and (33) into the Heisenberg equations of motion (17), using the following equation [5]

$$\hat{a}_i^m (\hat{a}_i^\dagger)^n = \sum_{j=0}^{min(m,n)} j! \binom{m}{j} \binom{n}{j} (\hat{a}_i^\dagger)^{n-j} \hat{a}_i^{m-j}, \quad i = 1, 2, \tag{34}$$

which can be proved easily using the commutation rules (1), and comparing the expressions corresponding to the same process, we obtain the following *infinite* set of differential equations for amplitudes $f_{MmR}, M = 0, 1, ...; m = [\frac{M}{2}] + 1, ..., M + 1;$ $R = [\frac{M+1}{2}] + m, ..., M + m$ and $g_{MmR}, M = 0, 1, ...;\ m = [\frac{M+1}{2}] + 1, ..., M + 2;$ $R = [\frac{M+1}{2}] + m, ..., M + m$:

$$\begin{aligned}\frac{d}{dt} f_{MmR}(t) &= 2i\Gamma \sum_{M_1, M_2=0}^{\infty} \sum_{m_1=[\frac{M_1}{2}]+1}^{M_1+1} \sum_{m_2=[\frac{M_2+1}{2}]+1}^{M_2+2} \sum_{R_1=[\frac{M_1+1}{2}]+m_1}^{M_1+m_1} \sum_{R_2=[\frac{M_2+1}{2}]+m_2}^{M_2+m_2} \\ &\quad \times \binom{2R_1 - M_1 - 2m_1}{s_1} \binom{2R_2 - M_2 - 2m_2}{s_1} \binom{M_1 + m_1 - R_1}{s_2} \\ &\quad \times \binom{M_2 + m_2 - R_2}{s_2} s_1! s_2! f^*_{M_1 m_1 R_1}(t) g_{M_2 m_2 R_2}(t), \end{aligned} \tag{35}$$

$$\begin{aligned}\frac{d}{dt} g_{MmR}(t) &= i\Gamma^* \sum_{M_1, M_2=0}^{\infty} \sum_{m_1=[\frac{M_1}{2}]+1}^{M_1+1} \sum_{m_2=[\frac{M_2}{2}]+1}^{M_2+1} \sum_{R_1=[\frac{M_1+1}{2}]+m_1}^{M_1+m_1} \sum_{R_2=[\frac{M_2+1}{2}]+m_2}^{M_2+m_2} \\ &\quad \times \binom{2m_1 - M_1 - 1}{s_1} \binom{2R_2 - M_2 - 2m_2}{s_1} \binom{M_1 - m_1 + 1}{s_2} \\ &\quad \times \binom{M_2 + m_2 - R_2}{s_2} s_1! s_2! f_{M_1 m_1 R_1}(t) f_{M_2 m_2 R_2}(t), \end{aligned} \tag{36}$$

where

$$\begin{aligned} R_2 &= R + R_1 - 2m - 2m_1 + 2m_2, \\ s_1 &= M - M_1 - M_2 - 2m - 2m_1 + 2m_2 + 2R_1 - 1, \\ s_2 &= M_1 + M_2 - M - m - m_1 + m_2 + R - R_2 + 1 \end{aligned} \tag{37}$$

for equation (35) and

$$
\begin{aligned}
R_2 &= R - R_1 - 2m + 2m_1 + 2m_2, \\
s_1 &= M - M_1 - M_2 + 2m - 2m_1 - 2m_2 - 2R + 2R_1 + 2R_2, \\
s_2 &= M_1 + M_2 - M - m + m_1 + m_2 + R - R_1 - R_2
\end{aligned} \tag{38}
$$

for equation (36). The initial conditions for equations (35) and (36) are $f_{011}(0) = g_{011}(0) = 1, f_{MmR}(0) = g_{MmR}(0) = 0$ in all other cases.

The structure of R.H.S. of (35) reveals that there is a nonzero contribution to the R.H.S. only if the following inequalities hold simultaneously

$$
\begin{aligned}
2R_1 - M_1 - 2m_1 &\geq s_1, \quad 2R_2 - M_2 - 2m_2 \geq s_1, \\
M_1 + m_1 - R_1 &\geq s_2, \quad M_2 + m_2 - R_2 \geq s_2.
\end{aligned} \tag{39}
$$

Combining this with (37) one finally obtains

$$
M_1 \leq M - 1, \quad M_2 \leq M - 1. \tag{40}
$$

Analogously, R.H.S. of (36) contains nonzero contribution only if

$$
\begin{aligned}
2m_1 - M_1 - 1 &\geq s_1, \quad 2R_2 - M_2 - 2m_2 \geq s_1, \\
M_1 - m_1 + 1 &\geq s_2, \quad M_2 + m_2 - R_2 \geq s_2,
\end{aligned} \tag{41}
$$

hold simultaneously. Consequently, the R.H.S. of (36) cannot contain amplitudes other than those for which

$$
M_1 \leq M, \quad M_2 \leq M + 1. \tag{42}
$$

From the above inequalities (40) and (42) follows that for fixed M we have only *finite* set of equations (35), (36) for amplitudes $f_{M'm'R'}, M' = 0, 1, ..., M;\ m' = [\frac{M'}{2}]+1, ..., M'+1;\ R' = [\frac{M'+1}{2}]+m', ..., M'+m'$ and $g_{M''m''R''}, M'' = 0, 1, ..., M-1;\ m'' = [\frac{M''+1}{2}] + 1, ..., M'' + 2;\ R'' = [\frac{M''+1}{2}] + m'', ..., M'' + m''$. In other words, the process in the oscillator 1(2) parametrized by $M' > M(M'' > M - 1)$ does not affect the processes in the oscillator 1(2) for which $M' \leq M(M'' \leq M - 1)$, as we wanted to prove.

The discussion of the structure of the set of equations (35), (36) can go even further. Since the amplitudes $f_{M'm'R'}, M' < M$ and $g_{M''m''R''}, M'' < M - 1$ can be calculated solving the set of equations (35), (36) corresponding to $M - 1$, in fact only the amplitudes $f_{Mm'R'}, m' = [\frac{M}{2}] + 1, ..., M + 1;\ R' = [\frac{M+1}{2}] + m', ..., M + m'$ and $g_{M-1m''R''}, m'' = [\frac{M}{2}]+1, ..., M+1;\ R'' = [\frac{M}{2}]+m'', ..., M+m''-1$ are mutually coupled. The amplitudes $f_{M'm'R'}, M' < M$ and $g_{M''m''R''}, M'' < M - 1$ then play the role of known coefficients and source terms and the set of differential equations corresponding to M is linear. Moreover, taking M and m fixed, substituting (37) into (39) and putting $M_1 = M_2 = M - 1$ one obtains the following equality

$$
m_2 = m. \tag{43}
$$

Since the same equality can be proved substituting (38) into (41) and putting $M_1 = M, M_2 = M + 1$, one can conclude that only amplitudes $f_{MmR'}, R' = [\frac{M+1}{2}] + m, ..., M + m$ and $g_{M-1mR''}, R'' = [\frac{M}{2}] + m, ..., M + m - 1$ are coupled. Hence for given M and m one has to solve the set of $M + 1$ differential equations (35) and (36).

Before going further let us notice that since the infinite series (32) and (33) with amplitudes being the solutions of the equations (35) and (36) satisfy the Heisenberg equations of motion (17) identically, the operators $\hat{a}_1(t)$ and $\hat{a}_2(t)$ preserve the commutation rules (1).

Someone still could object that our method cannot be used in practice since we are not able to calculate all the amplitudes. This difficulty is, however, overcome if one realizes the following fact. Calculating the matrix element of the process corresponding to parameters M, m and R for oscillator 1,

$$\begin{aligned}&\langle N_1-2l_1,l_1|(\hat{a}_1^\dagger)^{2R-M-2m}(\hat{a}_2^\dagger)^{M+m-R}\hat{a}_1^{2m-M-1}\hat{a}_2^{M-m+1}|N_2-2l_2,l_2\rangle\\ &=\sqrt{\frac{l_1!l_2!(N_1-2l_1)!(N_2-2l_2)!}{(N_1-2l_1+M+2m-2R)!(N_2-2l_2+M-2m+1)!}}\\ &\times\frac{\delta_{N_1-2l_1+4m,N_2-2l_2+2R+1}\delta_{l_1+R+1,l_2+2m}}{\sqrt{(l_1-M-m+R)!(l_2-M+m-1)!}},\end{aligned}\tag{44}$$

where the formulas (5) and (8) have been used, it is evident that it does not vanish only if the following inequalities are satisfied simultaneously

$$\begin{aligned}N_1-2l_1 &\geq 2R-M-2m, \quad 2l_1\geq 2M+2m-2R,\\ N_2-2l_2 &\geq 2m-M-1, \quad 2l_2\geq 2M-2m+2.\end{aligned}\tag{45}$$

Hence

$$N_1\geq M, \quad N_2\geq M+1.\tag{46}$$

Repeating the same discussion for the same matrix element of the process in oscillator 2 characterized by the parameters M, m and R one arrives at

$$N_1\geq M, \quad N_2\geq M+2.\tag{47}$$

The inequalities (46) and (47) can be interpreted as follows. Restricting ourselves to the finite-dimensional subspace $\mathcal{H}_K$ of the whole Hilbert space (7) only processes in oscillator 1(2) for which $M'=0,1,...,K-1(M''=0,1,...,K-2)$ are represented by nonzero matrix. In other words, the time evolution of the operators $\hat{a}_1(t)$ and $\hat{a}_2(t)$ on the subspace $\mathcal{H}_K$ is known once the amplitudes $f_{M'm'R'}$, $M'=0,1,...,K-1$; $m'=[\frac{M'}{2}]+1,...,M'+1$; $R'=[\frac{M'+1}{2}]+m',...,M'+m'$ and $g_{M''m''R''}$, $M''=0,1,...,K-2$; $m''=[\frac{M''+1}{2}]+1,...,M''+2$; $R''=[\frac{M''+1}{2}]+m'',...,M''+m''$ are determined. This requires sequential solution of $[\frac{M}{2}]+1$ sets of $M=0,1,...,K$ differential equations (35) and (36). Since the series (32) and (33) are terminated naturally when considering only finite-dimensional subspace $\mathcal{H}_K$, a natural question arises whether the last-named amplitudes determine not only the evolution of $\hat{a}_1(t)$ and $\hat{a}_2(t)$ but also the evolution of any operator on the subspace $\mathcal{H}_K$. Now we will prove that this is really the case.

It is well known that any operator at time t on the space $\mathcal{H}$ can be expressed as a sum of the following products $(\hat{a}_1^\dagger)^i(t)(\hat{a}_2^\dagger)^j(t)\hat{a}_1^k(t)\hat{a}_2^l(t)$. Hence, it is sufficient to prove the statement for these products only. The commutation rules (1) enable us to show that

$$\begin{aligned}&\hat{N}(\hat{a}_1^\dagger)^{2R-M-2m}(\hat{a}_2^\dagger)^{M+m-R}\hat{a}_1^{2m-M-1}\hat{a}_2^{M-m+1}|N-2l,l\rangle\\ &=(N-1)(\hat{a}_1^\dagger)^{2R-M-2m}(\hat{a}_2^\dagger)^{M+m-R}\hat{a}_1^{2m-M-1}\hat{a}_2^{M-m+1}|N-2l,l\rangle\end{aligned}\tag{48}$$

for oscillator 1 and similarly

$$\begin{aligned}&\hat{N}(\hat{a}_1^\dagger)^{2R-M-2m}(\hat{a}_2^\dagger)^{M+m-R}\hat{a}_1^{2m-M-2}\hat{a}_2^{M-m+2}|N-2l,l\rangle\\ &=(N-2)(\hat{a}_1^\dagger)^{2R-M-2m}(\hat{a}_2^\dagger)^{M+m-R}\hat{a}_1^{2m-M-2}\hat{a}_2^{M-m+2}|N-2l,l\rangle\end{aligned}\tag{49}$$

for oscillator 2. Consquently,

$$\hat{N}\hat{a}_j(t)|N-2l,l\rangle = (N-j)\hat{a}_j(t)|N-2l,l\rangle, \quad j=1,2,\tag{50}$$

as one can verify, using (32) and (33). From that it follows that $\hat{a}_j(t)\mathcal{H}_K \subset \mathcal{H}_{K-j}$, $j = 1, 2$. Since the first annihilation (creation) operator to the right (left) in the matrix elements

$$\langle N_1 - 2l_1, l_1|(\hat{a}_1^\dagger)^i(t)(\hat{a}_2^\dagger)^j(t)\hat{a}_1^k(t)\hat{a}_2^l(t)|N_2 - 2l_2, l_2\rangle, \quad N_1, N_2 = 0, 1, ..., K \tag{51}$$

transforms the basis vector to the right (left) into the subspace embeded into $\mathcal{H}_K$, the series (32) and (33) for the following annihilation (creation) operators must terminate even further than those for the first annihilation (creation) operator. Therefore no other amplitude except for those mentioned above can appear in the expression (51). This is what we wanted to prove.

There is one more point connected with the previous discussion which should be clarified here. Namely, one could think about the finite series for $\hat{a}_1(t)$ and $\hat{a}_2(t)$ on the $\mathcal{H}_K$ as an approximate operator solutions of (17). The following special example disproves the idea.

Let us consider the following finite series (describing correctly the time evolution on the subspace $\mathcal{H}_2$)

$$\begin{aligned} \hat{a}_1(t) &= f_{011}(t)\hat{a}_1 + f_{112}(t)\hat{a}_1^\dagger\hat{a}_2 + f_{123}(t)\hat{a}_1^\dagger\hat{a}_1^2, & (52)\\ \hat{a}_2(t) &= g_{011}(t)\hat{a}_2 + g_{022}(t)\hat{a}_1^2. & (53) \end{aligned}$$

The amplitudes in (52) and (53) are solutions of the set of equations

$$\begin{aligned} \frac{d}{dt}f_{011}(t) &= 0, \\ \frac{d}{dt}f_{112}(t) &= 2i\Gamma f_{011}^*(t)g_{011}(t), \quad \frac{d}{dt}g_{011}(t) = i\Gamma^* f_{011}(t)f_{112}(t), \\ \frac{d}{dt}f_{123}(t) &= 2i\Gamma f_{011}^*(t)g_{022}(t), \quad \frac{d}{dt}g_{022}(t) = i\Gamma^* f_{011}(t)(f_{011}(t) + f_{123}(t)), \end{aligned} \tag{54}$$

with the initial conditions $f_{011}(0) = g_{011}(0) = 1$, $f_{112}(0) = f_{123}(0) = g_{022}(0) = 0$. The use of standard methods then yields

$$\begin{aligned} f_{011}(t) &= 1, \\ f_{112}(t) &= i\frac{\sqrt{2}\Gamma}{|\Gamma|}\sin(\sqrt{2}|\Gamma|t), \quad g_{011}(t) = \cos(\sqrt{2}|\Gamma|t), \\ f_{123}(t) &= \cos(\sqrt{2}|\Gamma|t) - 1, \quad g_{022}(t) = i\frac{\Gamma^*}{\sqrt{2}|\Gamma|}\sin(\sqrt{2}|\Gamma|t). \end{aligned} \tag{55}$$

Now, substituting (55) into (53) one arrives at

$$[\hat{a}_2(t), \hat{a}_2^\dagger(t)] = \hat{1} + 2\sin^2(\sqrt{2}|\Gamma|t)\hat{a}_1^\dagger\hat{a}_1 \neq \hat{1}. \tag{56}$$

In the same way we can prove that $[\hat{a}_1(t), \hat{a}_1^\dagger(t)] \neq \hat{1}$. This illustrates that the commutation rules are not preserved for the finite series (52) and (53). One can expect this, since the series (52) and (53) represent a correct solution only on the subspace $\mathcal{H}_2$. Thus only the complete solution (32) and (33) involving all the processes preserves the commutation rules in the whole Hilbert space.

5 Conclusion

On the simple example we illustrate how to solve the nonlinear Heisenberg equations of motion on the finite-dimensional subspace using the finite expansion of annihilation operators into the sum of the elementary processes. The idea of the method is not restricted to this example and provides us with recipe how to treat other nonlinear interactions. The time evolution of any operator on the subspace is then governed by a finite number of the c-number differential equations for amplitudes. Due to the hierarchy of the processes the equations split into several sets which can be solved step by step. Thus the problem of solution of the q-number Heisenberg equations is transformed into the finding of solution of the linear c-number differential equations, which can be handled numerically. It provides a nice interpretation and deeper insight into what happens in the course of the nonlinear quantal interaction in the language of elementary processes. It also enables us to identify the non-classical contributions. This instructive interpretation cannot be obtained within the framework of the Schrödinger picture.

6 Acknowledgments

We would like to thank Prof. J. Peřina for useful advices. This work was supported by Grant LN00A015 of Czech Ministry of Education.

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